

BOREL'S CONJECTURE IN TOPOLOGICAL GROUPS

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ABSTRACT. We introduce a natural generalization of Borel's Conjecture. For each infinite cardinal number κ , let BC_κ denote this generalization. Then BC_{\aleph_0} is equivalent to the classical Borel conjecture. We obtain the following consistency results:

- (1) If it is consistent that there is a 1-inaccessible cardinal then it is consistent that BC_{\aleph_1} holds.
- (2) If it is consistent that BC_{\aleph_1} holds, then it is consistent that there is an inaccessible cardinal.
- (3) If it is consistent that there is a 1-inaccessible cardinal with ω inaccessible cardinals above it, then $\neg\text{BC}_{\aleph_\omega} + (\forall n < \omega)\text{BC}_{\aleph_n}$ is consistent.
- (4) If it is consistent that there is a 2-huge cardinal, then it is consistent that $\text{BC}_{\aleph_\omega}$ holds.
- (5) If it is consistent that there is a 3-huge cardinal, then it is consistent that BC_κ holds for a proper class of cardinals κ of countable cofinality.

A metric space (X, d) is *strong measure zero* if there is for each sequence $(\epsilon_n : n < \omega)$ of positive real numbers a corresponding sequence $(U_n : n < \omega)$ of open sets such that for each n the set U_n has d -diameter at most ϵ_n , and $\{U_n : n < \omega\}$ covers X . Strong measure zero metric spaces are necessarily separable. E. Borel [5] conjectured that strong measure zero sets of real numbers are countable. The metric notion of strong measure zero has a natural generalization to non-metric contexts. Rothberger [15] introduced a generalization to the class of topological spaces. We consider a generalization to the class of topological groups. Most of our results can be presented in the more general context of uniformizable spaces, but we found no advantage to presenting it thus.

Borel's Conjecture also has natural generalizations to these non-metric contexts. These generalizations expose, as in the metric case, interesting connections with the foundations of mathematics. The generalization of Borel's Conjecture considered here is quite different from what Halko and Shelah considered in [10].

After a brief introduction of notation and terminology we define *Rothberger boundedness* and *Rothberger spaces* in Section 1. In Section 2 we introduce a generalization of Borel's Conjecture and in Section 3 explore connections between it and other combinatorial structures. In Section 4 we give a number of consistency and independence results regarding the generalization introduced in the earlier sections.

By a well-known theorem of Kakutani a topological group is \mathcal{T}_0 if, and only if, it is $\mathcal{T}_{3\frac{1}{2}}$ ¹. Throughout this paper we shall assume, without further notice, that all groups considered are $\mathcal{T}_{3\frac{1}{2}}$. Correspondingly, all topological spaces we consider here are assumed to be $\mathcal{T}_{3\frac{1}{2}}$.

Let $(G, *)$ be a topological group with identity element $\mathbf{1}$. For nonempty subsets A and B of G and for $g \in G$ the symbol $A * B$ denotes the set $\{a * b : a \in A \text{ and } b \in B\}$, and $g * A$ denotes $\{g\} * A$. The symbol \mathcal{O} denotes the set of all nonempty open covers of G .

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¹A \mathcal{T}_0 topological group need not be normal.

Let U be an open neighborhood of $\mathbf{1}$. Then $\mathcal{O}(U) = \{g * U : g \in G\}$ is an open cover of G . Define:

$$\mathcal{O}_{nbd} := \{\mathcal{O}(U) : U \text{ an open neighborhood of } \mathbf{1}\}.$$

According to Guran [9] the topological group $(G, *)$ is \aleph_0 -bounded if each element of \mathcal{O}_{nbd} has a countable subset which covers G . A topological group is said to be *pre-compact* if each element of \mathcal{O}_{nbd} has a finite subset covering the group.

While pre-compact topological groups and Lindelöf topological groups are \aleph_0 -bounded the converse is not true. The class of \aleph_0 -bounded groups has nice preservation properties: Every subgroup of an \aleph_0 -bounded group is \aleph_0 -bounded, any (finite or infinite) product of \aleph_0 -bounded groups is \aleph_0 -bounded, every continuous homomorphic image of an \aleph_0 -bounded group is \aleph_0 -bounded, and if a dense subgroup of a group is \aleph_0 -bounded, then so is the group.

Theorem 1 (Guran). *A topological group is \aleph_0 -bounded if, and only if, it embeds as a topological group into a product of second countable topological groups.*

By the Birkhoff-Kakutani theorem each second countable T_0 topological group is metrizable. Thus the groups appearing as factors in the product in Guran's theorem are separable and metrizable. Guran's theorem has the following quantified form (see [23]):

Theorem 2. *For an \aleph_0 -bounded topological group $(G, *)$ and an infinite cardinal number κ the following are equivalent:*

- (1) *The weight of G is κ .*
- (2) *The character of G is κ .*
- (3) *κ is the smallest infinite cardinal such that G embeds as a topological group into a product of κ separable metrizable topological groups.*

1. ROTHBERGER BOUNDEDNESS IN TOPOLOGICAL GROUPS, ROTHBERGER SPACES.

For collections \mathcal{A} and \mathcal{B} the symbol $\mathbf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle

For each sequence $(A_n : n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n , $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

A topological space is said to be a *Rothberger space* if it has the property $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$. The topological group $(G, *)$ is said to be *Rothberger bounded* if it has the property $\mathbf{S}_1(\mathcal{O}_{nbd}, \mathcal{O})$. For a subset X of the topological group $(G, *)$ \mathcal{O}_X denote the family of covers of X by sets open in G . X is said to be Rothberger bounded if $\mathbf{S}_1(\mathcal{O}_{nbd}, \mathcal{O}_X)$ holds. These concepts are named after Rothberger who introduced $\mathbf{S}_1(\mathcal{O}, \mathcal{O})$ and who considered a close analogue of this boundedness property in the *Hilfssatz* on page 51 of his paper [15].

If a subset of a topological group is a Rothberger space, then it is Rothberger bounded in the group. The converse is not true. A subspace of a Rothberger space need not be a Rothberger space, but subsets of Rothberger bounded sets are Rothberger bounded. Rothberger boundedness of a subset of a group is preserved by continuous group homomorphisms and countable unions. The property of being a Rothberger space is preserved by continuous surjections and by countable unions.

Proposition 3. *Every Rothberger bounded subset of an \aleph_0 -bounded topological group is zero-dimensional.*

Proof: Let $(G, *)$ be an \aleph_0 -bounded group. Choose by Guran's theorem separable metrizable groups $(G_i, *_i)$, $i \in I$ such that $(G, *)$ embeds as topological group in the product $\prod_{i \in I} (G_i, *_i)$ and let Φ be an embedding. The projection of $\Phi[G]$ on each coordinate of

this product is a metrizable group. The Rothberger boundedness of subsets of G is also preserved by the composition of Φ and projections.

In metrizable groups Rothberger boundedness coincides with strong measure zero. Thus a Rothberger bounded subset X of an \aleph_0 -bounded T_0 group projects onto a metrizable strong measure set in each coordinate projection. By a theorem of Marczewski [22] strong measure zero metrizable spaces are zero-dimensional. Thus, as products and subspaces of zero-dimensional spaces are zero-dimensional, X is zero-dimensional. \square

The proof of Proposition 3 shows: Borel's Conjecture implies that each Rothberger bounded subset of an \aleph_0 -bounded topological group embeds as a Rothberger bounded subset into a product of countable topological groups (see [6] Theorem 3.2). It also follows that Rothberger spaces must be zero-dimensional, and that Borel's Conjecture implies that Rothberger spaces embed into products of countable topological groups.

If $C \subseteq \kappa$ is nonempty and if S is a subset of $\prod_{\alpha < \kappa} G_\alpha$, then $S_C = \{f \upharpoonright_C : f \in S\}$.

Lemma 4. *Let κ be an infinite cardinal number. Let $(G_\alpha : \alpha < \kappa)$ be topological groups and let a subset X of $G = \prod_{\alpha < \kappa} G_\alpha$ be given. The following are equivalent:*

- (1) *X is Rothberger bounded.*
- (2) *For each countable set $C \subseteq \kappa$ the set X_C is a Rothberger bounded subset of G_C .*

Proof: (1) \Rightarrow (2): A continuous group homomorphism preserves Rothberger boundedness.

(2) \Rightarrow (1): Suppose for each countable $C \subseteq \kappa$ we have that X_C is a Rothberger bounded subset of G_C . Let $(U_n : n < \omega)$ be a sequence of neighborhoods of the identity element of G . We may assume that each U_n is a basic open set such that there is a finite set $F_n \subseteq \kappa$ and for each $x \in F_n$ a neighborhood N_x of 1_x such that $U_n = \{f \in G : (\forall x \in F_n)(f(x) \in N_x)\}$. Let C be a countably infinite subset of κ for which $\bigcup_{n < \omega} F_n \subseteq C$. Then for each n , $V_n = U_n \upharpoonright_C \subseteq G_C$ is a neighborhood of the identity element of G_C . Since X_C is Rothberger bounded choose for each n an $x_n \in G_C$ such that $X_C \subseteq \bigcup_{n < \omega} x_n * V_n$. For each n choose $y_n \in G$ such that $y_n \upharpoonright_C = x_n$. Then we have $X \subseteq \bigcup_{n < \omega} y_n * U_n$. It follows that X is Rothberger bounded. \square

In the case of Rothberger spaces Lemma 4 has the following analogue:

Lemma 5. *Let κ be an infinite cardinal number. Let $(G_\alpha : \alpha < \kappa)$ be topological groups and let a subspace X of $G = \prod_{\alpha < \kappa} G_\alpha$ be given. Then the following are equivalent:*

- (1) *X is a Rothberger space.*
- (2) *X is Lindelöf and for each countable set $C \subseteq \kappa$ the set X_C is a Rothberger subspace of G_C .*

A Rothberger bounded sigma-compact subset of a topological group is a Rothberger space: First note that a union of countably many Rothberger subspaces of a space is again a Rothberger subspace. Thus, it suffices to show that a compact Rothberger bounded subset of a topological group is a Rothberger space. For this, use of the following analogue of the Lebesgue covering Lemma, following from Theorem 6.33 in [11]:

Lemma 6. *Let C be a compact subset of a topological group $(G, *)$ and let \mathcal{U} be a cover of C by sets open in G . Then there is a neighborhood N of the identity of G such that for each $x \in C$ there is a $U \in \mathcal{U}$ such that $x * N \subseteq U$.*

Corollary 7. *For each infinite cardinal κ any topological group $\prod_{\alpha < \kappa} G_\alpha$, where each G_α has at least two elements, has a Rothberger subgroup² of cardinality κ .*

²That is, a subgroup which is a Rothberger space.

Proof: For each $\alpha < \kappa$, choose a nontrivial (meaning that it has more than one element) countable subgroup H_α of G_α , and let $H = \prod_{\alpha < \kappa} H_\alpha$. Then $X = \{x \in H : x_\alpha = 1 \text{ for all but finitely many } \alpha\}$ is a subgroup of H of cardinality κ . Inasmuch as X is Lindelöf (in fact sigma-compact), and X_C is countable whenever C is countable, it follows by Lemma 5 that X is a Rothberger space. \square

Thus there are Rothberger (and so Rothberger bounded) groups of all infinite cardinalities.

2. THE GENERALIZED BOREL CONJECTURE

Let BC denote Borel's conjecture that each strong measure zero set of real numbers is countable³. For the real line with the addition operation, Borel's conjecture translates to the statement that every Rothberger bounded subset of the topological group $(\mathbb{R}, +)$ is countable.

Theorem 8. *The following statements are equivalent:*

- (1) BC
- (2) *Each strong measure zero metric space is countable.*
- (3) *Each Rothberger bounded subset of a group of countable weight is countable.*
- (4) *Each subgroup⁴ of the real line, all of whose finite powers are Rothberger, is countable.*

Proof: (1) \Leftrightarrow (2): This result is due to T.J. Carlson [6].

(2) \Rightarrow (3): By the Kakutani-Birkhoff Theorem a T_0 group of countable weight is metrizable by a left-invariant metric. Any Rothberger bounded subset X of such a group is strong measure zero in such a left-invariant metric. By 2), X is countable.

(3) \Rightarrow (4): Consider a subgroup G of the real line such that each finite power of G is Rothberger. Since G is Rothberger it is Rothberger bounded. Since the real line is a second countable group, 3) implies that G is countable.

(4) \Rightarrow (1): If X is a set of real numbers, then $[X]$, the subgroup of the real line generated by X , is a countable union of continuous images of finite powers of X , and the same goes for any finite power of $[X]$. Hence, if X has the property that all of its finite powers are Rothberger, then $[X]$ also has that property. Thus the implication (4) \Rightarrow (1) follows from the result of Tsaban and Weiss [25] that BC is equivalent to the statement that each subset of the real line, all of whose finite powers are Rothberger, is countable. \square

For λ a cardinal number and $(G, *)$ a topological group, $BC(G, < \lambda)$ states:

Each Rothberger bounded subset of $(G, *)$ has cardinality less than λ .

$BC(G, < \lambda^+)$ is also written $BC(G, \lambda)$ and $BC(G, \omega)$ is also written $BC(G)$.

It is clear that if $\mu < \lambda$ then $BC(G, < \mu)$ implies $BC(G, < \lambda)$. Moreover, if H is a subgroup of the group G then $BC(G, < \mu)$ implies $BC(H, < \mu)$.

Corollary 7 shows that for each infinite cardinal κ the statement $BC(^{\kappa}2, < \kappa)$ is false. The status of $BC(G, \kappa)$ for \aleph_0 -bounded groups of weight κ is more elusive. For an infinite cardinal number κ we define, inspired by (3) of Theorem 8:

³Sierpiński [19] proved that the Continuum Hypothesis implies the negation of Borel's conjecture, and Laver [13] proved that Borel's conjecture is consistent relative to the consistency of ZFC, the Zermelo-Fraenkel axioms plus the Axiom of Choice. Thus, Borel's conjecture is independent of ZFC.

⁴(4) remains equivalent to BC if we change "subgroup" to "subfield".

BC_κ: Each Rothberger bounded subset of an \aleph_0 -bounded group of weight κ has cardinality at most κ .

Thus, BC_{\aleph_0} is Borel's Conjecture, **BC**. It is also clear that for each infinite cardinal κ , BC_κ implies $\text{BC}({}^\kappa 2, \kappa)$. The status of $\text{BC}({}^\kappa 2, \kappa)$ is the focus of this paper.

3. BC_κ FOR UNCOUNTABLE κ .

Let $\lambda \leq \kappa$ be uncountable cardinal numbers. A family \mathcal{F} of subsets of κ such that $|\mathcal{F}| > \kappa$ and for each infinite subset A of κ such that $|A| < \lambda$, we have $|\{X \cap A : X \in \mathcal{F}\}| \leq |A|$, is said to be a (κ, λ) Kurepa family⁵. The (κ, λ) Kurepa Hypothesis, $\text{KH}(\kappa, \lambda)$, is the assertion that there exists a (κ, λ) Kurepa family. $\text{KH}(\aleph_1, \aleph_1)$ is the classical Kurepa Hypothesis.

Theorem 9. *Let κ be an uncountable cardinal. Let $(G_\alpha : \alpha < \kappa)$ be a family of topological groups, each with more than one element. If $\text{KH}(\kappa, \aleph_1)$ holds, then $\prod_{\alpha < \kappa} G_\alpha$ has a Rothberger bounded subset (indeed, subgroup) of cardinality κ^+ .*

Proof: For each $\alpha < \kappa$ let $\mathbf{1}_\alpha$ denote the identity element of, and let g_α be any other element of the group G_α . Let \mathcal{F} be a (κ, \aleph_1) Kurepa family on κ . For each $X \in \mathcal{F}$ define $\phi_X \in \prod_{\alpha < \kappa} G_\alpha$ so that for each $\alpha < \kappa$

$$\phi_X(\alpha) = \begin{cases} \mathbf{1}_\alpha & \text{if } \alpha \notin X \\ g_\alpha & \text{if } \alpha \in X \end{cases}$$

Then $S = \{\phi_X : X \in \mathcal{F}\}$ is a subset of cardinality $|\mathcal{F}|$ of $\prod_{\alpha < \kappa} G_\alpha$.

For each countable subset C of κ , the set $S_C = \{\phi_X \upharpoonright C : X \in \mathcal{F}\}$ has the same cardinality as $\{X \cap C : X \in \mathcal{F}\}$ and thus is countable. But then for each countable set $C \subset \kappa$, S_C is a Rothberger, and thus Rothberger bounded, subset of $\prod_{\alpha \in C} G_\alpha$. By Lemma 4 S is a Rothberger bounded subset of $\prod_{\alpha < \kappa} G_\alpha$. Evidently S generates a Rothberger bounded group. \square

Corollary 10. *For uncountable cardinals κ , $\text{BC}({}^\kappa 2, \kappa)$ implies the failure of $\text{KH}(\kappa, \aleph_1)$.*

Solovay proved that Kurepa's Hypothesis is consistent (it holds in the Constructible Universe L). Silver [20] proved that the negation of Kurepa's Hypothesis is consistent if, and only if, it is consistent that there is an inaccessible cardinal. Thus, the consistency of BC_{\aleph_1} (or even $\text{BC}({}^{\omega_1} 2, \aleph_1)$) requires the consistency of the existence of an inaccessible cardinal.

Theorem 11. *For an uncountable cardinal κ , each of the statements below implies all the succeeding ones; moreover, if **BC** holds, then they are all equivalent.*

- (1) BC_κ holds.
- (2) $\text{BC}({}^\kappa 2, \kappa)$ holds.
- (3) Each Rothberger bounded subgroup of the group $({}^\kappa 2, \oplus)$ has cardinality at most κ .
- (4) $\text{KH}(\kappa, \aleph_1)$ fails.

Proof: It is clear that (1) implies (2) and that (2) implies (3). Theorem 9 shows that (3) implies (4). To see that (4) implies (1), assume that BC_{\aleph_0} holds and BC_κ fails. Let G be an \aleph_0 -bounded group of weight κ witnessing the failure of BC_κ , and let $S \subseteq G$ be a Rothberger bounded subset of G of cardinality κ^+ . By Theorem 2 choose separable metrizable groups G_α , $\alpha < \kappa$ such that G embeds as a topological group into $\prod_{\alpha < \kappa} G_\alpha$. Then S , considered a subset of $\prod_{\alpha < \kappa} G_\alpha$, is Rothberger bounded in the latter and of cardinality κ^+ . Thus, by Lemma 4, for each countable set $C \subseteq \kappa$ the set $S_C \subseteq \prod_{\alpha \in C} G_\alpha$ is Rothberger bounded.

⁵This definition is like the one in Chapter VII.3 of [7], but we do not require κ to be regular.

Since $\prod_{\alpha \in C} G_\alpha$ is a separable metrizable space and BC_{\aleph_0} holds, Theorem 8 implies that S_C is countable. Considering S as a family of subsets of $\bigcup S$ we find that S is a witness that the statement $\text{KH}(\kappa, \aleph_1)$ is true. \square

Corollary 12. *Assume BC_{\aleph_0} . Then $\text{BC}^{(\omega_1 2, \aleph_1)}$ is equivalent to the failure of the Kurepa Hypothesis.*

A family \mathcal{F} of subsets of an uncountable cardinal κ is said to be a κ -Kurepa family if $|\mathcal{F}| > \kappa$ and for each infinite $\alpha < \kappa$ we have $|\{X \cap \alpha : X \in \mathcal{F}\}| \leq |\alpha|$. Kurepa's Hypothesis for κ , KH_κ , states that there exists a κ -Kurepa family. Note that a (κ, κ) -Kurepa family is also an example of a κ -Kurepa family. Thus, $\text{KH}(\kappa, \kappa)$ implies KH_κ ⁶. It is clear that $\text{KH}(\kappa, \kappa)$ also implies $\text{KH}(\kappa, \lambda)$ for each uncountable $\lambda < \kappa$.

Lemma 13. *Let κ be an uncountable cardinal. If $\text{KH}(\lambda, \lambda)$ fails for each uncountable $\lambda \leq \kappa$, then $\text{KH}(\kappa, \aleph_1)$ fails.*

Theorem 14. *For an uncountable cardinal κ , each of the statements below implies all the succeeding ones. If BC holds, then statements (1)-(3) are equivalent. If BC_λ holds for each infinite cardinal $\lambda < \kappa$, then all four statements are equivalent:*

- (1) BC_κ holds.
- (2) $\text{BC}^{(\kappa 2, \kappa)}$ holds.
- (3) $\text{KH}(\kappa, \aleph_1)$ fails.
- (4) $\text{KH}(\kappa, \kappa)$ fails.

Proof: In light of Theorem 11 and the definitions, the only statement that requires proof is: For each uncountable cardinal κ , if for each infinite $\lambda < \kappa$, BC_λ holds, then (4) implies (3). This will be proven by induction on κ .

For $\kappa = \aleph_1$ there is nothing to prove. Thus, assume that $\kappa > \aleph_1$ and that the statement has been proven for all uncountable cardinals less than κ . Towards proving the contrapositive, assume that $\text{KH}(\kappa, \aleph_1)$ holds. Let \mathcal{F} be a family of subsets of κ witnessing $\text{KH}(\kappa, \aleph_1)$. Then $|\mathcal{F}| > \kappa$, and for each countable subset A of κ , $|\{X \cap A : X \in \mathcal{F}\}| \leq \aleph_0$. Since BC_λ holds for each infinite cardinal $\lambda < \kappa$, Theorem 11 implies that $\text{KH}(\lambda, \aleph_1)$ fails for each uncountable $\lambda < \kappa$. By the induction hypothesis, $\text{KH}(\lambda, \lambda)$ fails for each uncountable cardinal $\lambda < \kappa$. Thus, for each uncountable ordinal $\alpha < \kappa$ we have $|\{X \cap \alpha : X \in \mathcal{F}\}| \leq |\alpha|$. But this means $\text{KH}(\kappa, \kappa)$ holds. \square

Now consider inaccessible cardinals of uncountable cofinality. An uncountable regular cardinal κ is *ineffable* if there is for each sequence $(A_\alpha : \alpha < \kappa)$ where for each α , $A_\alpha \subseteq \alpha$, a set $A \subseteq \kappa$ for which $\{\alpha < \kappa : A_\alpha = A \cap \alpha\}$ is stationary.

Theorem 15. *Let κ be an ineffable cardinal. If BC_λ holds for each infinite cardinal $\lambda < \kappa$, then BC_κ holds.*

Proof: Let κ be an ineffable cardinal. Then $\text{KH}(\kappa, \kappa)$ fails ([7], Theorem VII.3.1). By Theorem 14 BC_κ holds. \square

An increasing sequence $(\nu_\alpha : \alpha < \mu)$ of cardinals is said to be continuous if for each limit ordinal $\beta < \mu$ we have $\nu_\beta = \sup\{\nu_\alpha : \alpha < \beta\}$.

Theorem 16. *Let κ be a singular strong limit cardinal of uncountable cofinality μ . If there is an increasing continuous μ -sequence of cardinal numbers $(\nu_\alpha : \alpha < \mu)$ below κ with supremum equal to κ such that $\{\alpha < \mu : \text{BC}_{\nu_\alpha} \text{ holds}\}$ is a stationary subset of μ , then BC_κ holds.*

⁶We don't know if the converse is true.

Proof: Let κ be a singular strong limit cardinal of uncountable cofinality μ . Let S be a Rothberger bounded subset of an \aleph_0 -bounded group G of weight κ . By Theorem 2 we may assume that G embeds as topological group in the product $\prod_{\alpha < \kappa} G_\alpha$ where each G_α is a separable metrizable group. Let $(\nu_\alpha : \alpha < \mu)$, an increasing continuous μ -sequence of cardinal numbers with supremum equal to κ , be such that $\{\alpha < \mu : \text{BC}_{\nu_\alpha} \text{ holds}\}$ is a stationary subset of μ .

For each $\alpha < \mu$ the set $S_\alpha = \{f \upharpoonright_{\nu_\alpha} : f \in S\}$ is Rothberger bounded in the \aleph_0 -bounded group $\prod_{\beta < \nu_\alpha} G_\beta$ of weight at most ν_α . By hypothesis the set $\{\alpha < \mu : |S_\alpha| \leq \nu_\alpha\}$ is stationary.

Theorem 6 of [8] implies that the pairwise disjoint family $(S_\alpha : \alpha < \mu)$ has at most κ almost disjoint transversals. Since distinct elements of S specify distinct almost disjoint transversals of $(S_\alpha : \alpha < \mu)$, it follows that $|S| \leq \kappa$. \square

Next we explore the relevance of Chang's Conjecture to instances of BC_κ . Consider a countable language \mathbf{L} with a distinguished unary relation symbol \mathbf{U} . We say that a structure \mathfrak{A} of \mathbf{L} is of type (κ, λ) if the underlying set A of \mathfrak{A} has cardinality κ , and $\{x \in A : \mathbf{U}^{\mathfrak{A}}(x)\}$ has cardinality λ .

For infinite cardinal numbers κ, λ, μ and ν the symbol

$$(1) \quad (\kappa, \lambda) \twoheadrightarrow (\mu, \nu)$$

denotes the statement that for each countable language \mathbf{L} with a distinguished unary relation symbol \mathbf{U} , and for each structure \mathfrak{A} of type (κ, λ) there is an elementary substructure \mathfrak{B} of type (μ, ν) . The instances of interest have $\kappa > \lambda, \mu > \nu, \kappa \geq \mu$ and $\lambda > \nu$. The instance $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ is the classical conjecture of Chang.

Rowbottom [16] discovered a convenient combinatorial equivalent for (1): For infinite cardinal numbers κ, λ, μ and ν the symbol

$$(2) \quad \kappa \rightarrow [\mu]_{\lambda, \nu}^{< \aleph_0}$$

denotes the statement that for each function f from $[\kappa]^{< \aleph_0}$, the set of finite subsets of κ , into λ , there is a set $X \subseteq \kappa$ such that $|X| = \mu$, and $|\{f(Y) : Y \text{ is a finite subset of } X\}| \leq \nu$. The following lemma, a special case of a theorem of Rowbottom, is stated in the form we will use.

Lemma 17 (Rowbottom). *Let $\kappa > \lambda$ be infinite cardinal numbers. Then $(\kappa^+, \kappa) \twoheadrightarrow (\lambda^+, \lambda)$ is equivalent to $\kappa^+ \rightarrow [\lambda^+]_{\kappa, \lambda}^{< \aleph_0}$.*

For infinite cardinal numbers κ, λ, μ and ν the symbol

$$(3) \quad \kappa \rightarrow [\mu]_{\lambda, \nu}^2$$

denotes the statement that for each function f from $[\kappa]^2$, the set of 2-element subsets of κ , into λ , there is a set $X \subseteq \kappa$ such that $|X| = \mu$, and $|\{f(Y) : Y \subseteq X, |Y| = 2\}| \leq \nu$.

It is evident that $\kappa \rightarrow [\mu]_{\lambda, \nu}^{< \aleph_0}$ implies $\kappa \rightarrow [\mu]_{\lambda, \nu}^2$. When λ is a regular cardinal the converse is also true. A proof of this fact can be gleaned from the corresponding argument for $\kappa = \aleph_2, \lambda = \mu = \aleph_1$ and $\nu = \aleph_0$ on page 592 of [4].

Theorem 18. *Assume that for the infinite cardinal numbers κ and λ the partition relation $\kappa^+ \rightarrow [\lambda^+]_{\kappa, \lambda}^2$ holds. Then BC_λ implies BC_κ , and $\text{BC}(\lambda^2, \lambda)$ implies $\text{BC}(\kappa^2, \kappa)$.*

Proof: Suppose, towards deriving a contradiction, that BC_κ fails. Select an \aleph_0 -bounded group $(G, *)$ of weight κ and a subset X of G such that X is Rothberger bounded and $|X| = \kappa^+$. By Theorem 2 there are separable metrizable groups $(G_\alpha : \alpha < \kappa)$ such that G is

a subgroup of $\prod_{\alpha < \kappa} G_\alpha$, and X is a subset of this product. Define a coloring Φ from $[X]^2$ to κ so that

$$\Phi(\{f, g\}) = \min\{\gamma < \kappa : f(\gamma) \neq g(\gamma)\}.$$

Apply the partition relation to this coloring to find a subset Y of X and a subset C of κ such that $|Y| = \lambda^+$ and $|C| = \lambda$ and Φ restricted to $[Y]^2$ has values all in C . Y_C is Rothberger bounded since the projection map is a continuous homomorphism, and $|Y_C| = \lambda^+$ since Φ is one-to-one on Y . But then the group G_C contains a λ^+ -sized Rothberger bounded set Y_C , and as $G_C \subseteq \prod_{\alpha \in C} G_\alpha$ this \aleph_0 -bounded group has weight at most λ . This provides a contradiction to \mathbf{BC}_λ .

The proof that $\mathbf{BC}(\lambda^2, \lambda)$ implies $\mathbf{BC}(\kappa^2, \kappa)$ is left to the reader. \square

4. CONSISTENCY RESULTS

We now consider the consistency of instances of the general Borel Conjecture.

4.1. Consistency of the total failure of the general Borel Conjecture.

Theorem 19. *If $(G, *)$ is an \aleph_0 -bounded (totally bounded) topological group and $(\mathbb{P}, <)$ is a forcing notion, then*

$$\mathbf{1}_{\mathbb{P}} \Vdash “(\check{G}, *) \text{ is } \aleph_0\text{-bounded (respectively totally bounded)}”.$$

Proof: Let \dot{U} be a \mathbb{P} -name such that $\mathbf{1}_{\mathbb{P}} \Vdash “\dot{U} \text{ is a neighborhood of the identity}”$. Choose a maximal antichain A for \mathbb{P} and for each $q \in A$ a neighborhood U_q of the identity such that $q \Vdash “\dot{U} = \check{U}_q”$. Since $(G, *)$ is \aleph_0 -bounded, choose for each $q \in A$ a countable set $X_q := \{x_n^q : n < \omega\}$ of elements of G such that $X_q * U_q = G$. Define $\check{X} = \{(\check{x}_n^q, q) : n < \omega \text{ and } q \in A\}$. Then \check{X} is a \mathbb{P} -name and

$$\mathbf{1} \Vdash “\check{X} \subseteq \check{G} \text{ is countable and } \check{X} * \dot{U} = \check{G}”$$

Thus, $\mathbf{1}_{\mathbb{P}} \Vdash “(\check{G}, *) \text{ is } \aleph_0\text{-bounded}”$. The argument for totally bounded is similar. \square

Theorem 20. *If $(G, *)$ is an \aleph_0 -bounded group then in generic extensions by uncountably many Cohen reals, $(G, *)$ is Rothberger bounded.*

Proof: Let $(\mathbb{P}(\kappa), <)$ denote the partially ordered set for adding $\kappa > \aleph_0$ Cohen reals. By Theorem 19 $\mathbf{1}_{\mathbb{P}(\kappa)} \Vdash “(\check{G}, *) \text{ is } \aleph_0\text{-bounded}”$. Let $(\dot{\mathcal{U}}_n : n < \omega)$ be a name for a sequence of elements of \mathcal{O}_{nbd} . Since $\mathbb{P}(\kappa)$ has the countable chain condition and κ is uncountable, there is a countable subset C of κ such that $(\dot{\mathcal{U}}_n : n < \omega)$ is a $\mathbb{P}(C)$ -name. As the forcing factors over C we may assume that in fact the sequence so named is a ground model sequence. Since $(G, *)$ is \aleph_0 -bounded in this model also, we may select for each n a countable set $X_n \subset G$ such that $G = X_n * U_n$. For each x define for each n , $f_x(n) = m$ if $x \in x_m * U_n$, $x_m \in X_n$. These objects are all in the ground model. Take a Cohen real over the ground model. It selects a sequence of elements of G which witness Rothberger boundedness. \square

Theorem 21. *It is consistent, relative to the consistency of ZFC, that $\mathbf{BC}(\kappa^2, \kappa)$ fails for each infinite cardinal number κ .*

Proof: In the model of Theorem 20, for each infinite cardinal κ the ground model version of the additive group κ^2 is a Rothberger bounded group of cardinality 2^κ . \square

Since adding \aleph_1 Cohen reals leaves large cardinal properties of the ground model intact, there is no large cardinal property that implies any instance of $\mathbf{BC}(\kappa^2, \kappa)$.

4.2. Consistency of $\text{BC}_{\aleph_0} + \text{BC}_{\aleph_1}$. A partially ordered set $(\mathbb{P}, <)$ is said to have the *Laver property* if for each $h \in {}^\omega\omega$ it is forced that whenever τ , a term in the forcing language of \mathbb{P} , is such that $(\forall n)(\tau(n) < \check{h}(n))$ then there exists an $f \in {}^\omega[\omega]^{<\omega}$ such that $(\forall n)(|f(n)| \leq 2^n)$ and $(\forall n)(\tau(n) \in \check{f}(n))$, and for all but finitely many n , $f(n) \subseteq h(n)$.

If in the generic extensions obtained from a partially ordered set all the real numbers are members of the ground model, then the partially ordered set has the Laver property by default.

The importance of the Laver property is twofold:

Lemma 22 (Shelah). *A countable support iteration of partially ordered sets, each satisfying the Laver property, satisfies the Laver property⁷.*

The second important fact about the Laver property is the following folklore result for which a proof can be found in [2], Lemma 3.1:

Lemma 23. *Let X be a set of real numbers which does not have strong measure zero. If $(\mathbb{P}, <)$ is a partially ordered set with the Laver property, then*

$$\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\check{X} \text{ does not have strong measure zero.”}$$

A cardinal κ is said to be *1-inaccessible* if it is inaccessible, and there are κ many inaccessible cardinal numbers less than κ . And now we obtain the following consistency result:

Theorem 24. *If it is consistent that there is a 1-inaccessible cardinal, then it is consistent that ZFC plus Borel's Conjecture plus the negation of Kurepa's Hypothesis, plus $2^{\aleph_1} = \aleph_2$ hold.*

Proof: Let $(\kappa_\alpha : \alpha < \kappa)$ be a monotonic enumeration of the inaccessible cardinals below κ . We construct a κ -stage countable support iteration \mathbb{P}_κ as follows: Let $\pi : \kappa \rightarrow \kappa \times \kappa$ be a bookkeeping function such that

- For each $(\beta, \gamma) \in \kappa \times \kappa$ the set $\{\alpha < \kappa : \pi(\alpha) = (\beta, \gamma)\}$ is cofinal in κ ;
- If $\pi(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$.

\mathbb{P}_1 is defined as follows:

Let \mathbb{L}_0 denote the Levy collapse of κ_0 to ω_2 with countable conditions. By Silver's Theorem,

$$(4) \quad \mathbf{1}_{\mathbb{L}_0} \Vdash \text{“CH + There are no } \omega_1 \text{ Kurepa trees”}$$

Let $\langle \dot{T}_\gamma^0 : \gamma < \check{\kappa}_0 \rangle$ enumerate \mathbb{L}_0 -names of all ω_1 -trees with nodes members of ω_1 . By Silver's Theorem each has at most \aleph_1 cofinal branches. Pick $\pi(0) = (0, \gamma_0)$, and let $\dot{\mathbb{E}}_0$ be an \mathbb{L}_0 -name for a proper partially ordered set that does not add reals and specializes⁸ $\dot{T}_{\gamma_0}^0$ (see Chapter 5, Theorem 6.1 and Theorem 7.1 of [18]). Here we use the fact that if an ω_1 -tree has $\leq \aleph_1$ cofinal branches of length ω_1 , then it has a subtree with no cofinal ω_1 -branches, such that rendering this subtree special ensures that no further forcing that preserves ω_1 will add new cofinal ω_1 -branches through the original tree. Since $\mathbf{1}_{\mathbb{L}_0} \Vdash \text{“}\dot{\mathbb{E}}_0 \text{ has the Laver property”}$ it follows that $\mathbb{L}_0 * \dot{\mathbb{E}}_0$ has the Laver property. Next, let $\dot{\mathbb{M}}$ be a $\mathbb{L}_0 * \dot{\mathbb{E}}_0$ -name for the Mathias reals partially ordered set. Since the Mathias reals partially ordered set has the Laver property and forces that every uncountable ground-model set of reals does not have strong measure zero, we find that $\mathbb{L}_0 * \dot{\mathbb{E}}_0 * \dot{\mathbb{M}}$ has the Laver property

⁷See Conclusion 2.12 in Chapter VI.2 of [18].

⁸In the sense of Baumgartner - see Section 8 of [3].

and forces that CH holds and every uncountable set of reals from its ground model fails to be strong measure zero. We set $\mathbb{P}_1 = \mathcal{Q}_0 = \mathbb{L}_0 * \dot{\mathbb{E}}_0 * \dot{\mathbb{M}}$.

With $\alpha \leq \kappa$, and assuming that each \mathbb{P}_β has been defined for $\beta < \alpha$.

\mathbb{P}_α is defined as follows:

$\alpha = \beta + 1$ and $\beta \geq 1$: Define a \mathbb{P}_β name $\dot{\mathcal{Q}}_\beta$ for a partially ordered set as follows: Let $\dot{\mathbb{L}}_\beta$ be a \mathbb{P}_β name for the Levy collapse of κ_β to ω_2 with countable conditions. Let $\langle \dot{T}_\gamma^\beta : \gamma < \check{\kappa}_\beta \rangle$ enumerate $\dot{\mathbb{L}}_\beta$ -names for all ω_1 trees with nodes elements of ω_1 . With $\pi(\beta) = (\delta, \gamma)$ let $\dot{\mathbb{E}}_\beta$ be a $\dot{\mathbb{L}}_\beta$ -name for specializing the ω_1 tree \dot{T}_γ^δ (note that as $\delta \leq \beta$ the most recent Levy collapse ensures that this tree is not a Kurepa tree), and let $\dot{\mathbb{M}}$ be a $\dot{\mathbb{L}}_\beta * \dot{\mathbb{E}}_\beta$ -name for the Mathias reals partially ordered set. Finally we set

$$\dot{\mathcal{Q}}_\beta = \dot{\mathbb{L}}_\beta * \dot{\mathbb{E}}_\beta * \dot{\mathbb{M}} \text{ and } \mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathcal{Q}}_\beta.$$

Then we have

$$(5) \quad \mathbf{1}_{\mathbb{P}_\beta} \Vdash \text{“}\dot{\mathcal{Q}}_\beta \text{ has the Laver property”}$$

and also

$$(6) \quad \mathbf{1}_{\mathbb{P}_\beta} \Vdash \text{“}\dot{\mathbf{1}}_{\dot{\mathcal{Q}}_\beta} \Vdash \text{uncountable ground model sets of reals are not strong measure zero”}$$

α a limit ordinal: If α has countable cofinality then \mathbb{P}_α is the inverse limit of the $\mathbb{P}_\beta, \beta < \alpha$, and else \mathbb{P}_α is the direct limit of $\mathbb{P}_\beta, \beta < \alpha$.

Since κ is inaccessible, for each $\beta < \kappa$ $|\mathbb{P}_\beta| < \kappa$. Then \mathbb{P}_κ has the κ -chain condition. It also follows from Lemma 22 that $\mathbb{P}_\beta, \beta \leq \kappa$ has the Laver property.

To see that

$$\mathbf{1}_{\mathbb{P}_\kappa} \Vdash \text{“There are no } \omega_1 \text{ Kurepa trees”}$$

let

$$\mathbf{1}_{\mathbb{P}_\kappa} \Vdash \text{“}(\check{\omega}_1, \check{<}) \text{ is a tree order”}$$

Since \mathbb{P}_κ has the κ -chain condition and κ is inaccessible we find a $\beta < \kappa$ such that $(\check{\omega}_1, \check{<})$ is a \mathbb{P}_β name and $\mathbf{1}_{\mathbb{P}_\beta} \Vdash \text{“}(\check{\omega}_1, \check{<}) \text{ is a tree order”}$. But then

$$\mathbf{1}_{\mathbb{P}_\beta} \Vdash \text{“}\dot{\mathbf{1}}_{\dot{\mathbb{L}}_\beta} \Vdash \text{“}(\check{\omega}_1, \check{<}) \text{ is not a Kurepa tree””}$$

Now let \dot{T}_γ^β be the $\dot{\mathbb{L}}_\beta$ name for $(\check{\omega}_1, \check{<})$, and choose an $\alpha \geq \beta$ such that $\pi(\alpha) = (\beta, \gamma)$, and now consider $\mathbb{P}_{\alpha+1}$. Since $\dot{\mathbb{L}}_\alpha$ is a Levy collapse of an inaccessible cardinal Silver's Theorem implies that $(\check{\omega}_1, \check{<})$ is an ω_1 tree with no more than \aleph_1 cofinal ω_1 branches. Since $\pi(\alpha) = (\beta, \gamma)$, it follows that $\dot{\mathbb{E}}_\alpha$ specializes $(\check{\omega}_1, \check{<})$. Consequently,

$$\mathbf{1}_{\mathbb{P}_\kappa} \Vdash \text{“}(\check{\omega}_1, \check{<}) \text{ is not a Kurepa tree”}$$

To see that

$$\mathbf{1}_{\mathbb{P}_\kappa} \Vdash \text{“BC”}$$

let \dot{X} be a \mathbb{P}_κ name such that

$$\mathbf{1}_{\mathbb{P}_\kappa} \Vdash \text{“}\dot{X} \text{ is a set of real numbers of cardinality } \aleph_1 \text{”}$$

By the κ chain condition and the strong inaccessibility of κ choose a $\beta < \kappa$ such that \dot{X} is a \mathbb{P}_β name and

$$\mathbf{1}_{\mathbb{P}_\beta} \Vdash \text{“}\dot{X} \text{ is a set of real numbers of cardinality } \aleph_1 \text{”}$$

From the definition of $\dot{\mathcal{Q}}_\beta$ it is clear that

$$\mathbf{1}_{\mathbb{P}_\beta} \Vdash \text{“}\dot{\mathbf{1}}_{\dot{\mathcal{Q}}_\beta} \Vdash \text{“}\dot{X} \text{ is not strong measure zero””}$$

and thus

$$\mathbf{1}_{\mathbb{P}_{\beta+1}} \Vdash \text{“}\dot{X} \text{ is not strong measure zero”}$$

Since $\mathbb{P}_{[\beta+2, \kappa)}$ has the Laver property it follows that

$$\mathbf{1}_{\mathbb{P}_\kappa} \Vdash \text{“}\dot{X} \text{ is not strong measure zero.”}$$

We leave to the reader the standard argument that in the generic extension we have $2^{\aleph_0} = \aleph_2 = \kappa$ \square

Corollary 25. *If it is consistent that there is an inaccessible cardinal which is a limit of inaccessible cardinals, then it is consistent that $\text{BC}_{\aleph_0} + \text{BC}_{\aleph_1}$ holds.*

Proof: Corollary 12 and Theorem 24. \square

Since we may assume the ground model is \mathbf{L} , we may assume that the generic model of Theorem 24 satisfies: For each uncountable cardinal κ , $2^\kappa = \kappa^+$ holds. It is well-known that $2^{\aleph_0} = \aleph_1$ implies $\neg \text{BC}_{\aleph_0}$. Theorem 24 shows that $2^{\aleph_1} = \aleph_2$ does not imply $\neg \text{BC}_{\aleph_1}$.

4.3. Consistency of $(\forall n < \omega) \text{BC}_{\aleph_n}$.

Lemma 26. *Let κ and λ be uncountable cardinal numbers with $\lambda < \kappa$. Let $(\mathbb{P}, <)$ be a partially ordered set which is κ^+ -closed. If $\neg \text{KH}_\lambda$ holds, then $\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\neg \text{KH}_\lambda \text{ holds.”}$*

Proof: This follows from Theorem VII.6.14 of [12]. \square

Lemma 27. *Let κ be a regular cardinal number with $2^{\aleph_0} \leq \kappa$. Let $(\mathbb{P}, <)$ be a partially ordered set which is κ -closed. If BC_{\aleph_0} holds, then $\mathbf{1}_{\mathbb{P}} \Vdash \text{“}\text{BC}_{\aleph_0} \text{ holds.”}$*

Proof: This also follows from Theorem VII.6.14 of [12]: No new sets of real numbers of cardinality \aleph_1 are added by this forcing. \square

Theorem 28. *Assume it is consistent that the following three statements hold: BC_{\aleph_0} , $\neg \text{KH}_{\aleph_1}$, $2^{\aleph_1} = \aleph_2$, and there are inaccessible cardinals $\kappa_0 < \dots < \kappa_n < \dots$, $n < \omega$. Then it is consistent that $\text{BC}_{\aleph_0} + (\forall n < \omega)(0 < n \Rightarrow \neg \text{KH}_{\aleph_n})$ holds.*

Proof: As in Exercise (F4) on p. 295 of [12] define a countable support iterated forcing poset \mathbb{P} such that successively “for each $n[> 0]$, κ_n is Levy collapsed to \aleph_{n+2} by conditions of cardinality $\leq \kappa_{n-1}$ ”. Then, by [3], Theorem 2.5, $(\mathbb{P}, <)$ is \aleph_2 -closed. By Lemmas 26 and 27 $(\mathbb{P}, <)$ preserves $\text{BC}_{\aleph_0} + \neg \text{KH}_{\aleph_1}$. By the cited exercise from [12], in the resulting generic extension we have $(\forall n < \omega)(0 < n \Rightarrow \neg \text{KH}_{\aleph_n})$. \square

Corollary 29. *If it is consistent that $\text{BC}_{\aleph_0} + \neg \text{KH}_{\aleph_1} + 2^{\aleph_1} = \aleph_2$ + there are inaccessible cardinals $\kappa_0 < \dots < \kappa_n < \dots$, $n < \omega$ holds, then it is consistent that $(\forall n < \omega) \text{BC}_{\aleph_n}$ holds.*

Proof: Theorem 28 and Theorem 14. \square

4.4. Consistency of BC_κ first failing at $\kappa = \aleph_\omega$.

Theorem 30. *Suppose is consistent that there is an inaccessible cardinal κ such that there are κ inaccessible cardinals below κ , and ω inaccessible cardinals above κ . Then it is consistent that $(\forall n < \omega) \text{BC}_{\aleph_n}$ while also $\neg \text{BC}_{\aleph_\omega}$.*

Proof: We may assume the ground model is \mathbf{L} . Let λ be the limit of the inaccessible cardinals assumed to exist in the hypothesis. Thus, λ has countable cofinality and there is a Kurepa family on λ . Performing the forcing in Theorem 24, followed by the forcing in Theorem 28 preserves this Kurepa family, but collapses λ to \aleph_ω . \square

4.5. Consistency of $\text{BC}_{\aleph_\omega}$.

An uncountable cardinal number κ is said to be μ -strong if there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $V_\mu \subseteq M$. κ is said to be a *strong cardinal* if it is μ -strong for all μ .

Theorem 31. *If it is consistent that for an uncountable cardinal κ of countable cofinality both $2^\kappa = \kappa^+$ and $\text{BC}(\kappa^2, \kappa)$ hold, then it is consistent that there is a strong cardinal.*

Proof: Todorcevic proved (see for example Chapter 7 of [24]) that if κ is an uncountable cardinal of countable cofinality then \square_κ plus $\text{cof}([\kappa]^{\aleph_0}, \subseteq) = \kappa^+$ implies that there is a cofinal in $[\kappa]^{\aleph_0}$ family of countable sets that witnesses $\text{KH}(\kappa, \aleph_1)$. Applying Theorem 11 we find that $\neg \text{BC}(\kappa^2, \kappa)$ holds.

Thus, if $\text{BC}(\kappa^2, \kappa)$ and $2^\kappa = \kappa^+$ hold, then \square_κ fails. Jensen has proved that failure of \square_κ for uncountable κ of countable cofinality implies the existence of an inner model with a strong cardinal. \square

In consistency strength strong cardinals lie between measurable cardinals and strongly compact cardinals: A strong cardinal is measurable. If κ is strongly compact then \square_λ fails for each cardinal $\lambda > \kappa$ and thus there is an inner model with a strong cardinal.

Theorem 32. *If $\text{BC}(\kappa^2, \kappa)$ holds for an uncountable cardinal κ of countable cofinality for which we have $\lambda^{\aleph_0} < \kappa$ for all $\lambda < \kappa$, then the axiom of projective determinacy is true.*

Proof: [17] Corollary 6. \square

Corollary 33. *If $2^{\aleph_0} < \aleph_\omega$ and if $\text{BC}(\aleph_\omega^2, \aleph_\omega)$ holds, then Projective Determinacy holds.*

Corollary 34. *If for each $n < \omega$ we have $2^{\aleph_n} < \aleph_\omega$ and if $\text{BC}(\aleph_\omega^2, \aleph_\omega)$ holds, then Determinacy holds in $\mathbb{L}(\mathbb{R})$.*

Proof: [21], Theorem 0.1. \square

Now we determine upper bounds on the consistency strength of $\text{BC}_{\aleph_\omega}$.

Lemma 35. *Let $\kappa > \lambda$ be infinite cardinal numbers. Let $(\mathbb{P}, <)$ be a κ^{++} -closed partially ordered set. If the partition relation $\kappa^+ \rightarrow [\lambda^+]^2_{\kappa, \lambda}$ holds, then*

$$(7) \quad \mathbf{1}_{\mathbb{P}} \Vdash \check{\kappa}^+ \rightarrow [\check{\lambda}^+]^2_{\check{\kappa}, \check{\lambda}}.$$

Proof: Assume that (7) is false. Choose $p \in \mathbb{P}$ such that $p \Vdash \check{\kappa}^+ \nrightarrow [\check{\lambda}^+]^2_{\check{\kappa}, \check{\lambda}}$. Choose a \mathbb{P} -name \dot{f} such that $p \Vdash \dot{f} : [\check{\kappa}^+]^2 \rightarrow \check{\lambda}^+$ witnesses $\check{\kappa}^+ \nrightarrow [\check{\lambda}^+]^2_{\check{\kappa}, \check{\lambda}}$. Since \mathbb{P} is κ^{++} -closed we find a ground model function $g : [\kappa^+]^2 \rightarrow \lambda^+$ and a $q < p$ such that $q \Vdash \dot{f} = \check{g}$ on the ground model set $[\kappa^+]^2$. Applying the partition relation $\kappa^+ \rightarrow [\lambda^+]^2_{\kappa, \lambda}$ in the ground model to g we find an uncountable ground model subset S of κ^+ and a ground model subset $C \subset \lambda^+$ such that $|C| \leq \lambda$ and for any $F \in [S]^2$ we have $g(F) \in C$. But then $q \Vdash \text{“For each finite } F \in [\check{S}]^2, \dot{f}(F) \in \check{C}\text{”}$, contradicting the fact that $q < p$ and $p \Vdash \check{\kappa}^+ \nrightarrow [\check{\lambda}^+]^2_{\check{\kappa}, \check{\lambda}}$. \square

For an ordinal number α and for a cardinal number κ , the notation $\kappa^{+\alpha}$ denotes the α -th cardinal number larger than κ .

Lemma 36 (Levinski, Magidor, Shelah). *Let μ be an inaccessible cardinal. Let $(\mathbb{P}, <)$ be a partially ordered set such that $|\mathbb{P}| = \mu$ and \mathbb{P} has the μ -chain condition. If $(\mu^{+\omega+1}, \mu^{+\omega}) \rightarrow (\aleph_1, \aleph_0)$ holds, then $\mathbf{1}_{\mathbb{P}} \Vdash \check{(\mu}^{+\omega+1}, \check{\mu}^{+\omega}) \rightarrow (\check{\aleph}_1, \check{\aleph}_0)$.”*

Proof: See page 168 of [14]. \square

If μ is an inaccessible cardinal and θ is a regular cardinal with $\mu > \theta$, then $\text{Lv}(\mu, \theta)$ is the set of p such that p is a function with $\text{dom}(p) \subseteq \mu \times \theta$, $|p| < \theta$, and for all $(\alpha, \xi) \in \text{dom}(p)$ we have $p(\alpha, \xi) \in \alpha$. For p and q in $\text{Lv}(\mu, \theta)$ write $q < p$ if $p \subset q$. Then the partially ordered set $(\text{Lv}(\mu, \theta), <)$ is the *Levy collapse*. It is θ -closed, has the μ -chain condition, and has cardinality μ .

Lemma 37 (Levinski, Magidor, Shelah). *Let $\lambda > \mu$ be inaccessible cardinals. If $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\mu^{+\omega+1}, \mu^{+\omega})$ holds, then $\mathbf{1}_{(\text{Lv}(\mu^{+\omega}, \omega), <)} \Vdash “(\check{\lambda}^{+\omega+1}, \check{\lambda}^{+\omega}) \twoheadrightarrow (\aleph_1, \aleph_0)”$.*

Proof: See p. 168 of [14]. \square

Lemma 38. *Let μ be an inaccessible cardinal such that $(\mu^{+\omega+1}, \mu^{+\omega}) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds. Let $\alpha < \mu$ be an ordinal and let $\theta < \mu$ be the cardinal $\aleph_{\omega \cdot \alpha + 2}$. Then*

$$\mathbf{1}_{(\text{Lv}(\mu, \theta), <)} \Vdash “(\aleph_{\omega \cdot (\alpha+1)+1}, \aleph_{\omega \cdot (\alpha+1)}) \twoheadrightarrow (\check{\aleph}_1, \check{\aleph}_0)”$$

Proof: By Lemma 36, $\mathbf{1}_{(\text{Lv}(\mu, \theta), <)} \Vdash “(\check{\mu}^{+\omega+1}, \check{\mu}^{+\omega}) \twoheadrightarrow (\check{\aleph}_1, \check{\aleph}_0)”$. It is well-known that $\mathbf{1}_{(\text{Lv}(\mu, \theta), <)} \Vdash “|\check{\mu}| = \aleph_{\omega \cdot \alpha + 3}”$. Consequently, $\mathbf{1}_{(\text{Lv}(\mu, \theta), <)} \Vdash “|\check{\mu}^{+\omega+1}| = \aleph_{\omega \cdot \alpha + \omega + 1}”$ and $\mathbf{1}_{(\text{Lv}(\mu, \theta), <)} \Vdash “|\check{\mu}^{+\omega}| = \aleph_{\omega \cdot \alpha + \omega}”$ \square

For $0 < n < \omega$ the uncountable cardinal λ is said to be *n-huge* if there is an elementary embedding $j : V \rightarrow M$ to a transitive inner model M of the set theoretic universe V such that λ is the critical point of j , and setting $\kappa_0 = \lambda$ and $\kappa_{i+1} = j(\kappa_i)$ for $i < n$, we have ${}^{\kappa_n}M \subseteq M$. It can be shown that if λ is *n-huge* then, in the above notation, each of the cardinals κ_i , $0 \leq i \leq n$ is measurable.

Lemma 39 (Levinski, Magidor, Shelah). *If λ is a 2-huge cardinal, then*

$$(\kappa_1^{+\omega+1}, \kappa_1^{+\omega}) \twoheadrightarrow (\lambda^{+\omega+1}, \lambda^{+\omega}).$$

Theorem 40. *If it is consistent that there is a 2-huge cardinal, then it is consistent that $\text{BC}_{\aleph_\omega}$ holds.*

Proof: Let λ be a 2-huge cardinal and let j be an elementary embedding witnessing this. Put $\kappa = j(\lambda)$. It is known that κ is measurable and $\kappa > \lambda$. By Lemma 39 $(\kappa^{+\omega+1}, \kappa^{+\omega}) \twoheadrightarrow (\lambda^{+\omega+1}, \lambda^{+\omega})$.

Forcing first with $\mathbb{P}_0 = (\text{Lv}(\lambda^{+\omega}, \omega), <)$ we obtain by Lemma 37 a generic extension in which we have $(\kappa^{+\omega+1}, \kappa^{+\omega}) \twoheadrightarrow (\aleph_1, \aleph_0)$. Since κ is still measurable in this generic extension, it is an inaccessible limit of inaccessible cardinals. Now let \mathbb{P}_1 be the corresponding partially ordered set for Theorem 24. Then $|\mathbb{P}_1| = \kappa$, and \mathbb{P}_1 has the κ -chain condition.

Since $\mathbf{1}_{\mathbb{P}_1} \Vdash “\check{\kappa} = \aleph_2”$, Lemma 36 gives $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ in the generic extension. By Theorem 24 this generic extension also satisfies BC_{\aleph_0} and BC_{\aleph_1} . But then since BC_{\aleph_0} holds, Lemma 17 and Theorem 18 imply that $\text{BC}_{\aleph_\omega}$ holds in this generic extension. \square

By the facts in the table below $\text{BC}_{\aleph_{\omega+n}} + \text{BC}_{\aleph_{\omega+n+1}}$ holds in the model of Theorem 40 for each $n < \omega$.

The argument to prove Lemma 39 gives for all n:	Then the model of Theorem 40 gives:
$(\kappa_1^{+\omega+n+1}, \kappa_1^{+\omega+n}) \twoheadrightarrow (\lambda^{+\omega+n+1}, \lambda^{+\omega+n}).$	$(\aleph_{\omega+n+1}, \aleph_{\omega+n}) \twoheadrightarrow (\aleph_{n+1}, \aleph_n).$
$(\kappa_1^{+\omega(n+1)+1}, \kappa_1^{+\omega(n+1)}) \twoheadrightarrow (\lambda^{+\omega(n+1)+1}, \lambda^{+\omega(n+1)}).$	$(\aleph_{\omega(n+1)+1}, \aleph_{\omega(n+1)}) \twoheadrightarrow (\aleph_{\omega n+1}, \aleph_{\omega n}).$

4.6. Consistency of: For a proper class of κ with $cf(\kappa) = \aleph_0$, BC_κ .

For ordinal number α a cardinal number λ is said to be n -huge α times if there is for each ordinal $\beta < \alpha$ an elementary embedding j_β into a transitive inner model M_β such that each j_β witnesses that λ is n -huge, and when $\beta < \delta < \alpha$, then $j_\beta(\lambda) < j_\delta(\lambda)$.

Theorem 41 (Barbanel, Di Prisco, Tan). *If λ is $n + 1$ -huge, then there is a cardinal $\mu < \lambda$ such that μ is n -huge, and*

$$\{\alpha < \lambda : \text{There is an } n\text{-huge elementary embedding } j \text{ with } j(\mu) = \alpha\}$$

is a stationary subset of λ .

Corollary 42. *If λ is a 3-huge cardinal, then there is a 2-huge cardinal μ such that*

$$T = \{\alpha < \lambda : \alpha \text{ is measurable and } (\alpha^{+\omega+1}, \alpha^{+\omega}) \twoheadrightarrow (\mu^{+\omega+1}, \mu^{+\omega})\}$$

is a stationary subset of λ .

Proof: Lemma 39 and Theorem 41. \square

Corollary 43. *If it is consistent that there is a 3-huge cardinal, then it is consistent that BC_{\aleph_0} as well as BC_{\aleph_1} hold, and there is a proper class of cardinals κ such that $\omega = cf(\kappa)$, and BC_κ as well as BC_{κ^+} hold.*

Proof: Now let T be as in Corollary 42. Upon forcing with $(Lv(\mu^{+\omega}, \omega), <)$ we find that

$$T = \{\alpha < \lambda : \alpha \text{ is a measurable cardinal and } (\alpha^{+\omega+1}, \alpha^{+\omega}) \twoheadrightarrow (\aleph_1, \aleph_0)\}$$

Enumerate T in increasing order as $(\alpha_\xi : \xi < \lambda)$. Next we force with the poset of Theorem 24, using an iteration of length α_0 . In the resulting model we have $\alpha_0 = \aleph_2$ and for all $\xi > 0$, α_ξ is still measurable. Moreover we have for each ξ that $BC_{\alpha_\xi^{+\omega}}$ as well as $BC_{\alpha_\xi^{+\omega+1}}$ hold. Since λ is still measurable, V_λ is a model of ZFC, and in V_λ we have for each $0 < \xi < \lambda$ that $BC_{\alpha_\xi^{+\omega}}$ as well as $BC_{\alpha_\xi^{+\omega+1}}$ hold. \square

5. QUESTIONS

In Theorem 21 we showed that $(\forall \kappa)(\neg BC(^{\kappa}2, \kappa))$ holds in generic extensions by \aleph_1 Cohen reals.

Problem 1. *Does $V = L$ imply $(\forall \kappa)(\neg BC(^{\kappa}2, \kappa))$?*

In all our models of instances of BC_κ also BC_{\aleph_0} is true.

Problem 2. *Is it consistent that $BC(^{\kappa}2, \kappa)$ holds for some uncountable cardinal κ , while BC fails? What if $\kappa = \aleph_1$ or $\kappa = \aleph_\omega$?*

$BC(^{\kappa}2, \kappa)$ implies that every Rothberger subspace of $^{\kappa}2$ has cardinality at most κ . For $\kappa = \aleph_0$ the converse is true. This is not known for $\kappa > \aleph_0$.

Problem 3. *Is it for each infinite cardinal κ true that if each Rothberger subspace of $^{\kappa}2$ has cardinality at most κ , then $BC(^{\kappa}2, \kappa)$ holds?*

For each κ , BC_κ implies $BC(^{\kappa}2, \kappa)$. For $\kappa > \aleph_0$ it is not clear if the converse is true.

Problem 4. *Is it true that for each uncountable cardinal κ , $BC(^{\kappa}2, \kappa)$ implies BC_κ ?*

We obtained from the consistency of a large cardinal hypothesis the consistency of the statement that BC_κ holds for a proper class of cardinals κ (of countable cofinality).

Problem 5. *Is $\text{ZFC} + (\forall \kappa) \text{BC}({}^\kappa 2, \kappa)$ consistent relative to the consistency of any large cardinal axioms?*

Our findings indicate that $\text{BC}({}^{\aleph_\omega} 2, \aleph_\omega)$ has considerable consistency strength.

Problem 6. *What is the exact consistency strength of $\text{BC}({}^{\aleph_\omega} 2, \aleph_\omega)$?*

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